The Exponential Integral
Part II - A Mean-Reverting Revenue Model

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In this white paper we will build a revenue model that incorporates mean reversion, which is the tendancy for a short-term unsustainable revenue growth rate to revert to a long-term sustainable mean over time. In continuous time we use exponential functions. If the variable t represents time and the equation within the exponential is linear (i.e. constant through time) then the antiderivative of that exponential function is easy to calculate. An example of this type of function is the following...

$$\mathrm{Exp}\left\{-a\,t\right\} \text{ ...where... the antiderivative with respect to t} = \frac{1}{a}\,\mathrm{Exp}\left\{-a\,t\right\} \tag{1}$$

A problem occurs when the equation within the exponential is nonlinear (i.e. is mean-reverting) in which case an antiderivative cannot be calculated. An example of this type of function is the following...

$$\mathrm{Exp}\left\{a\,\mathrm{Exp}\left\{-b\,t\right\}\right\} \text{ ...where... the antiderivative with respect to t} = \text{Undefined} \tag{2}$$

To solve an integral using the example in Equation (2) above we map that function to the exponential integral via a change of variables and then solve the exponential integral, which we discussed in Part I of this series. To this end we will use the following hypothetical problem...

Our Hypothetical Problem

We are valuing a company where the short-term revenue growth rate is unsustainable and is expected to decrease to the long-term sustainable rate over a ten year period. We are tasked with building a revenue model to answer three questions...

Table 1: Model Assumptions

Symbol	Description	Value
R_0	Annualized revenue at time zero (in dollars)	1,000,000
T	Transition period in years	10.00
μ_0	Short-term annualized revenue growth rate (%)	25.00
μ_T	Long-term annualized revenue growth rate (%)	5.00

We will use our model to answer the following questions:

Question 1: What is the rate of mean reversion?
Question 2: What annualized revenue at the end of year three?
Question 3: What is cumulative revenue realized in year two?

The Mean-Reverting Revenue Growth Rate

We will define the variable μ_t to be the annualized revenue growth rate at time t, the variable μ_0 to be the current unsustainable revenue growth rate at time zero, and the variable λ to be the rate of mean reversion. The equation for the annualized revenue growth rate at time t is...

$$\mu_t = \mu_0\,\mathrm{Exp}\left\{-\lambda\,t\right\} \tag{3}$$

Note that one of the limitations of the equation above is that as time goes to infinity the revenue growth rate goes to zero rather than some constant and for that reason the equation above should only be used over a finite time interval. This statement in equation form is...

$$\lim_{t \to \infty} \mu_t = 0 \text{ ...because... } \lim_{t \to \infty} \mathrm{Exp}\Bigg\{ -\lambda\, t \Bigg\} = 0 \text{ ...when... } \lambda > 0 \tag{4}$$

Given the limit equation above, if we know what the annualized revenue growth rate will be at some time $T > 0$ (i.e. μ_T is a known value) then using Equation (3) above we can solve for the rate of mean reversion as follows...

$$\lambda = -\ln\left(\frac{\mu_T}{\mu_0}\right) \Bigg/ T \tag{5}$$

We will define the variable Γ_t to be the cumulative revenue growth rate realized over the time interval $[0, t]$. Using Equation (3) above the equation for the cumulative revenue growth rate at time t is...

$$\Gamma_t = \int_0^t \mu_t\, \delta t \tag{6}$$

Note that using Equation (3) above we can rewrite Equation (6) above as...

$$\Gamma_t = \int_0^t \mu_0\, \mathrm{Exp}\Bigg\{ -\lambda\, t \Bigg\}\, \delta t = \mu_0 \int_0^t \mathrm{Exp}\Bigg\{ -\lambda\, s \Bigg\}\, \delta s \tag{7}$$

Using Appendix Equation (32) below the solution to Equation (7) above is...

$$\Gamma_t = \mu_0 \Bigg(1 - \mathrm{Exp}\Bigg\{ -\lambda\, t \Bigg\} \Bigg) \lambda^{-1} \tag{8}$$

Annualized Revenue

We will define the variable R_t to be annualized revenue at time t. Using Equation (6) above the equation for annualized revenue is...

$$R_t = R_0\, \mathrm{Exp}\Bigg\{ \Gamma_t \Bigg\} \tag{9}$$

Note that using Equation (8) above we can rewrite Equation (27) above as...

$$R_t = R_0\, \mathrm{Exp}\Bigg\{ \mu_0 \Bigg(1 - \mathrm{Exp}\Bigg\{ -\lambda\, t \Bigg\} \Bigg) \lambda^{-1} \Bigg\} = R_0\, \mathrm{Exp}\Bigg\{ \mu_0\, \lambda^{-1} \Bigg\}\, \mathrm{Exp}\Bigg\{ -\mu_0\, \lambda^{-1} \mathrm{Exp}\Bigg\{ -\lambda\, t \Bigg\} \Bigg\} \tag{10}$$

We will make the following definitions...

$$a = -\mu_0\, \lambda^{-1} \text{ ...and... } b = \lambda \tag{11}$$

Using the definitions in Equation (11) above we can rewrite annualized revenue Equation (10) above as...

$$R_t = R_0\, \mathrm{Exp}\Bigg\{ -a \Bigg\}\, \mathrm{Exp}\Bigg\{ a\, \mathrm{Exp}\Bigg\{ -b\, t \Bigg\} \Bigg\} \tag{12}$$

We will define the variable $\bar{R}_{m,n}$ to be cumulative revenue realized over the time interval $[m, n]$. The equation for cumulative revenue is...

$$\bar{R}_{m,n} = \int_m^n R_s\, \delta s \text{ ...where... } m < n \tag{13}$$

Using Equation (12) above we can rewrite Equation (13) above as...

$$\bar{R}_{m,n} = R_0\, \mathrm{Exp}\Bigg\{ -a \Bigg\} \int_m^n \mathrm{Exp}\Bigg\{ a\, \mathrm{Exp}\Bigg\{ -b\, s \Bigg\} \Bigg\}\, \delta s \tag{14}$$

Note that the integral in Equation (14) above has an exponential function within an exponential function and therefore an antiderivative of the integrand cannot be found by normal means. If we define the variable I to be the integral in Equation (14) above then we can rewrite that equation as...

$$\bar{R}_{m,n}=R_0\,\mathrm{Exp}\Big\{-a\Big\}\,I\ \text{...where...}\ I=\int\limits_{s=m}^{s=n}\mathrm{Exp}\Big\{a\,\mathrm{Exp}\Big\{-b\,s\Big\}\Big\}\,\delta s \tag{15}$$

Our goal will be to map the integral above into the exponential integral and then solve it.

Solving The Integral

We will perform the following change of variables so as to integrate in terms of the new variable u rather than s...

$$\text{if...}\ u=a\,\mathrm{Exp}\Big\{-b\,s\Big\}\ \text{...then...}\ \frac{\delta u}{\delta s}=-b\,a\,\mathrm{Exp}\Big\{-b\,s\Big\}=-b\,u\ \text{...and...}\ \delta s=-\frac{1}{b}\,\frac{1}{u}\,\delta u \tag{16}$$

Since we are now integrating in terms of the new variable u then using the definitions in Equation (16) above we will change the bounds of integration for integral I above as follows...

$$\bar{m}=a\,\mathrm{Exp}\Big\{-b\,m\Big\}\ \text{...and...}\ \bar{n}=a\,\mathrm{Exp}\Big\{-b\,n\Big\} \tag{17}$$

Using Equations (16) and (17) above we can rewrite integral I in Equation (15) above as...

$$I=\int\limits_{u=\bar{m}}^{u=\bar{n}}\mathrm{Exp}\Big\{u\Big\}\times-\frac{1}{b}\times\frac{1}{u}\times\delta u \tag{18}$$

Note that in the integral above $\bar{m}$ is less than $\bar{n}$ and both terms are negative. If we map integral I to the exponential integral then the bounds of integration must be positive and the upper bound must be greater than the lower bound. We will therefore make the following change of variables...

$$v=-u\ \text{...where...}\ \frac{\delta v}{\delta u}\ \text{...such that...}\ \delta u=-\delta v \tag{19}$$

Using the definitions in Equation (19) we can rewrite Equation (18) above as...

$$I=\int\limits_{v=-\bar{m}}^{v=-\bar{n}}\mathrm{Exp}\Big\{-v\Big\}\times-\frac{1}{b}\times-\frac{1}{v}\times-\delta v=-\frac{1}{b}\int\limits_{v=-\bar{m}}^{v=-\bar{n}}\mathrm{Exp}\Big\{-v\Big\}\,\frac{1}{v}\,\delta v \tag{20}$$

Note that in Equation (20) above the integral's upper bound $(-\bar{n})$ is less than lower bound $(-\bar{m})$ and therefore we want to switch the bounds of integration. To do this we can rewrite Equation (20) above as...

$$I=-\frac{1}{b}\int\limits_{v=-\bar{n}}^{v=-\bar{m}}\mathrm{Exp}\Big\{-v\Big\}\,\frac{1}{v}\,\delta v\times-1=\frac{1}{b}\int\limits_{v=-\bar{n}}^{v=-\bar{m}}\mathrm{Exp}\Big\{-v\Big\}\,\frac{1}{v}\,\delta v \tag{21}$$

Using Equation (21) above we can rewrite cumulative revenue Equation (15) above as...

$$\bar{R}_{m,n}=R_0\,\mathrm{Exp}\Big\{-a\Big\}\,\frac{1}{b}\int\limits_{v=-\bar{n}}^{v=-\bar{m}}\mathrm{Exp}\Big\{-v\Big\}\,\frac{1}{v}\,\delta v \tag{22}$$

Using the equations from Part I we can rewrite cumulative revenue Equation (22) above as... [1]

$$\bar{R}_{m,n}=R_0\,\mathrm{Exp}\Big\{-a\Big\}\,\frac{1}{b}\Big(\mathrm{EXPINTU}(-\bar{n})-\mathrm{EXPINTU}(-\bar{m})\Big) \tag{23}$$

Answers To Our Hypothetical Problem

Question 1: What is the rate of mean reversion?

Using Table 1 above the revenue growth rates in continuous time are...

$$\mu_0 = \ln\left(1+0.25\right) = 0.2231 \text{ ...and... } \mu_T = \ln\left(1+0.05\right) = 0.0488 \tag{24}$$

Using Equation (5) above the equation for the rate of mean reversion is...

$$\lambda = -\ln\left(\frac{0.0488}{0.2231}\right)\Big/ 10.00 = 0.1520 \tag{25}$$

Question 2: What annualized revenue at the end of year three?

Using Equation (8) above the cumulative revenue growth rate over the time interval $[0, 3]$ is...

$$\Gamma_3 = 0.2231 \times \left(1 - \text{Exp}\left\{-0.1520 \times 3\right\}\right) \times 0.1520^{-1} = 0.5375 \tag{26}$$

Using Equation (27) above annualized revenue at the end of year 3 is...

$$R_3 = 1,000,000 \times \text{Exp}\left\{0.5375\right\} = 1,711,700 \tag{27}$$

Question 3: What is cumulative revenue realized in year two?

Using Equation (11) above the definition of exponential integral parameters a and b are...

$$a = -0.2231 \times 0.1520^{-1} = -1.4678 \text{ ...and... } b = 0.1520 \tag{28}$$

Using the parameters to our hypothetical problem the definition of the variables m and n are...

$$m = 1.0000 \text{ ...and... } n = 2.0000 \tag{29}$$

Using Equation (16) above the definition of exponential integral parameters $\bar{m}$ and $\bar{n}$ are...

$$\bar{m} = -1.4678 \times \text{Exp}\left\{-0.1520 \times 1.00\right\} = -1.2608 \text{ ...and... } \bar{n} = -1.4678 \times \text{Exp}\left\{-0.1520 \times 2.00\right\} = -1.0830 \tag{30}$$

Using cumulative revenue Equation (23) above the answer to the question is...

$$\begin{aligned}\bar{R}_{m,n} &= R_0 \times \text{Exp}\left\{1.4678\right\} \times \frac{1}{0.1520} \times \left(\text{EXPINTU}(1.0830) - \text{EXPINTU}(1.2608)\right)\\ &= 1,000,000 \times 4.3396 \times 6.5777 \times \left(0.1912 - 0.1440\right)\\ &= 1,349,000\end{aligned} \tag{31}$$

References

[1] Gary Schurman, *The Exponential Integral - Part I,* November, 2017

Appendix

A. The solution to the integral in Equation (7) above is...

$$\int_0^t \text{Exp}\left\{-\lambda\, s\right\} \delta s = -\frac{1}{\lambda}\text{Exp}\left\{-\lambda\, s\right\}\Big[_0^t = -\frac{1}{\lambda}\left(\text{Exp}\left\{-\lambda\, t\right\} - 1\right) = \left(1 - \text{Exp}\left\{-\lambda\, t\right\}\right)\lambda^{-1} \tag{32}$$